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# ON GRAPHS RELATED TO CONJUGACY CLASSES OF GROUPS*

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ABSTRACT

Let $G$ be a finite group. Attach to $G$ the following two graphs: $\Gamma$ — its vertices are the non-central conjugacy classes of $G$, and two vertices are connected if their sizes are *not* coprime, and $\Gamma^*$ — its vertices are the prime divisors of sizes of conjugacy classes of $G$, and two vertices are connected if they both divide the size of some conjugacy class of $G$. We prove that whenever $\Gamma^*$ is connected then its diameter is at most 3, (this result was independently proved in [3], for solvable groups) and $\Gamma^*$ is disconnected if and only if $G$ is quasi-Frobenius with abelian kernel and complements. Using the method of that proof we give an alternative proof to Theorems in [1],[2],[6], namely that the diameter of $\Gamma$ is also at most 3, whenever the graph is connected, and that $\Gamma$ is disconnected if and only if $G$ is quasi-Frobenius with abelian kernel and complements. As a result we conclude that both $\Gamma$ and $\Gamma^*$ have at most two connected components. In [2],[3] it is shown that the above bounds are best possible.

## Introduction and notation

Throughout this paper we shall use the following notation:

$G \equiv$ a finite group.

$S_p(G) \equiv$ the set of all Sylow p-subgroups of $G$.

$\mathrm{Con}(G) \equiv$ the set of conjugacy classes of $G$.

$\pi_i \equiv$ sets of primes.

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$\pi(n) \equiv$ the set of prime divisors of $n$.

$\pi(A) \equiv \pi(|A|)$ for any finite set $A$.

$\rho(G) \equiv \bigcup\{\ \pi(C) : C \in \mathrm{Con}(G)\}$.

$\Gamma(G) \equiv (V, E)$ — the graph whose set of vertices, $V$, are the non-central conjugacy classes of $G$ and two classes $A, B$ are joined by an edge $AB \in E$ if $(|A|, |B|) > 1$. For $AB \in E$ and an integer $n > 1$ we write $A \underset{n}{\leftrightarrow} B$ if both $|A|$ and $|B|$ are divisible by $n$.

$\Gamma^*(G) \equiv (V^*, E^*)$ — the dual graph whose set of vertices $V^* = \rho(G)$ and two distinct primes $p, q$ are joined by an edge $pq \in E^*$ if there is a class $C \in \mathrm{Con}(G)$ such that $pq \mid |C|$. For $pq \in E^*$ and $C \in \mathrm{Con}(G)$ we write $p \underset{C}{\leftrightarrow} q$ if both $p$ and $q$ divide $|C|$.

For $S \subseteq V$, let $\Gamma(G)\big|_S$ be the induced graph obtained by restricting $\Gamma(G)$ to the set $S$, i.e.

$$\Gamma(G)\big|_S \equiv (S, E \cap (S \times S)).$$

We sometimes speak of the **subgraph** $S$ meaning $\Gamma(G)\big|_S$.

For $a, b \in V$, let $d(a, b)$ denote the distance between $a$ and $b$ in $\Gamma(G)$ and for $S \subseteq V$ let $d(a, S) \equiv \min\{\ d(a, s) : s \in S\ \}$. Let $d(G)$ be the diameter of $\Gamma(G)$. Similarly define : $\Gamma^*(G)\big|_\Delta, d^*(p, q), d^*(p, \Delta)$ and $d^*(G)$, where $p, q \in V^*$ and $\Delta \subseteq V^*$.

A group $G$ is called **quasi-Frobenius** if $G/Z(G)$ is Frobenius. The inverse images in $G$ of the kernel and complements of $G/Z(G)$ are then called the **kernel** and **complements** of $G$.

It was shown in [6], and later rediscovered in [1] and [2], that $\Gamma(G)$ has diameter at most 3 whenever the graph is connected, and that $\Gamma(G)$ is disconnected if and only if $G$ is quasi-Frobenius with abelian kernel and complements. It also follows that $\Gamma(G)$ has at most two connected components.

In this paper we obtain some information about the structure of $\Gamma^*(G)$. Its vertices can be partitioned as follows : $V^* = \Delta \uplus \Lambda$, so that $\Gamma^*(G)\big|_\Delta$ is a complete graph and either every vertex of $\Lambda$ is connected to it by an edge, or $\Gamma^*(G)\big|_\Lambda$ is also a complete graph. If $\Gamma^*(G)$ is connected it follows that its diameter is at most 3 (this result was independently proved in [3], for solvable groups). It also follows that $\Gamma^*(G)$ has at most two connected components and we show that it is disconnected if and only if $G$ is quasi-Frobenius with abelian kernel and complements. We shall also use these proofs to give alternative proofs of the above mentioned results from [1],[2],[6]. It was shown in [2] and [3] that the above bounds are best possible.

## 1. Preliminaries

THEOREM 1.1 (Schur–Zassenhaus, Hall–Cunihin [7]):

(a) *A group with a normal Hall $\pi$-subgroup is $\pi$-separable.*

(b) *In a $\pi$-separable group:*

  (i) *Every subgroup and homomorphic image is $\pi$-separable.*

  (ii) *Every $\pi$-subgroup is contained in a Hall $\pi$-subgroup.*

  (iii) *All the Hall $\pi$-subgroups are conjugate.*

(c) *$\pi$-separability and $\pi'$-separability are equivalent.*

LEMMA 1.2 (Gorenstein [4]): *If $H, K$ are subgroups of $G$ of relatively prime indices, then $G = HK$ and $|G\colon H \cap K| = |G\colon H| \cdot |G\colon K|$.*

THEOREM 1.3 (N.Ito [5]): *For two distinct primes $p, q$ if $pq$ is not an edge of $\Gamma^*(G)$, then either $N_G(P) = C_G(P)$ or $N_G(Q) = C_G(Q)$ where $P \in S_p(G)$ and $Q \in S_q(G)$, and thus $G$ is either $p$- or $q$-nilpotent.*

LEMMA 1.4: *If $a, b \in G$ are commuting elements of relatively prime orders then: $C_G(ab) = C_G(a) \cap C_G(b)$, whence $|\mathrm{Cl}(a)|$ and $|\mathrm{Cl}(b)|$ divide $|\mathrm{Cl}(ab)|$.*

LEMMA 1.5: *For any $X \subseteq G$ and $g \in G$, $C_G(X^g) = C_G(X)^g$.*

LEMMA 1.6: *In any group $G$ : $d(G) \le d^*(G) + 1$ and $d^*(G) \le d(G) + 1$.*

*Proof:* Let $A, B \in \mathrm{Con}(G)$ and $p, q \in \rho(G)$ such that $p \mid |A|$ and $q \mid |B|$. Assume we have in $\Gamma(G)$ the path :

$$A = C_1 \underset{p_1}{\longleftrightarrow} C_2 \underset{p_2}{\longleftrightarrow} \cdots\cdots \underset{p_{l-1}}{\longleftrightarrow} C_l \underset{p_l}{\longleftrightarrow} C_{l+1} = B$$

where $C_i \in \mathrm{Con}(G)$ and $p_i \in \rho(G)$. Then in $\Gamma^*(G)$ we have the path :

$$p \underset{C_1}{\longleftrightarrow} p_1 \underset{C_2}{\longleftrightarrow} \cdots\cdots \underset{C_l}{\longleftrightarrow} p_l \underset{C_{l+1}}{\longleftrightarrow} q$$

if $p \neq p_1$ and $q \neq p_l$ (otherwise the path from $p$ to $q$ is even shorter). Thus we conclude that $d^*(p, q) \le d^*(A, B) + 1$ and it follows that $d^*(G) \le d(G) + 1$. Likewise, if we have in $\Gamma^*(G)$ the path :

$$p = p_1 \underset{C_1}{\longleftrightarrow} p_2 \underset{C_2}{\longleftrightarrow} \cdots\cdots \underset{C_{l-1}}{\longleftrightarrow} p_l \underset{C_l}{\longleftrightarrow} p_{l+1} = q$$

then in $\Gamma(G)$ we have the path :

$$A \underset{p_1}{\longleftrightarrow} C_1 \underset{p_2}{\longleftrightarrow} \cdots\cdots \underset{p_l}{\longleftrightarrow} C_l \underset{p_{l+1}}{\longleftrightarrow} B$$

and $d(G) \leq d^*(G) + 1$ follows. ∎

COROLLARY 1.7: *$\Gamma(G)$ is connected if and only if $\Gamma^*(G)$ is connected.*

LEMMA 1.8: *A group $G$, with a subgroup $H \neq 1, G$ for which: $H^x \cap H = \{1\}$ $\forall x \notin H$, is Frobenius with a complement $H$ and the kernel $K = \{1\} \cup (G \setminus \bigcup_{x \in G} H^x)$.*

LEMMA 1.9: *In any group $G$ : $\rho(G) = \pi(G/Z(G))$.*

*Proof:* Let $p$ be a prime number. Assume $p \notin \rho(G)$ and fix $P \in S_p(G)$. Let $x \in G$; since $p \nmid |\mathrm{Cl}(x)|$ it follows that $C_G(x)$ contains a Sylow $p$-subgroup of $G$, so there exists $g \in G$ such that $P^g \leq C_G(x)$, whence $x \in C_G(P^g) = C_G(P)^g$. Since this is true for every $x \in G$ we have $G = \cup_{g \in G} C_G(P)^g$, which implies that $C_G(P) = G$, so $P \leq Z(G)$ and $p \nmid |G/Z(G)|$.

Conversely, if $p \notin \pi(G/Z(G))$ then $G$ has a central Sylow $p$-subgroup $P$. So $P \leq C_G(x)$ for every $x \in G$, thus $p \nmid |\mathrm{Cl}(x)|$ and $p \notin \rho(G)$. ∎

If $G = A \times G_1$, where $A$ is abelian, then $\Gamma^*(G_1) = \Gamma^*(G)$ and $\Gamma(G_1)$ is similar to $\Gamma(G)$ with possibly more repetitions of vertices (i.e. more classes of the same size). So, while investigating the diameter and connectedness of $\Gamma(G)$ and $\Gamma^*(G)$, we can assume, without loss of generality, that $G$ has no abelian factors and therefore $\rho(G) = \pi(G)$ (since a central Sylow subgroup is an abelian direct factor).

Let

$$\Delta(G) \equiv \{\, p \in \pi(G) : N_G(P) \neq C_G(P) \text{ for every } P \in S_p(G) \,\};$$

and

$$\Lambda(G) \equiv \{\, p \in \pi(G) : N_G(P) = C_G(P) \text{ for every } P \in S_p(G) \,\}.$$

When there is no danger of confusion we shall omit $(G)$ and simply write: $\Delta$ and $\Lambda$.

By Theorem 1.3, $\Gamma^*(G)\big|_\Delta$ is a complete graph.

*Remark 1.10:* In any group $G$ : $\Delta(G) = \pi(G')$.

*Proof:* Assume $p \notin \Delta(G)$. Then by Burnside's Theorem, $G$ has a semidirect decomposition $G = PN$ where $P \in S_p(G)$ is abelian and $N \lhd G$. Since $G/N \cong P$ is abelian, $G' \leq N$ and $p \notin \pi(G')$. Conversely, if $p \notin \pi(G')$ and $P \in S_p(G)$, then $[P, N_G(P)] \leq P \cap G' = 1$, whence $N_G(P) = C_G(P)$ and $p \notin \Delta(G)$. ∎

COROLLARY 1.11: *If $\pi(G') = \pi(G)$, and in particular if $G$ is perfect, then $\Gamma^*(G)$ is complete.*

*Remark 1.12:* If $G$ is perfect, then $\Gamma(G)$ is not necessarily complete. For example, $G = A_5 \times \mathrm{PSL}(3,2)$ is perfect, but it has classes of sizes 20 and 21.

## 2. $\Gamma^*(G)$

LEMMA 2.1: *Suppose a group $G$ has two semidirect decompositions: $G = H_1N_1 = H_2N_2$ where $H_i$ are abelian Hall $\pi_i$-subgroups, $N_i$ are normal Hall $\pi_i'$-subgroups and $\pi_1 \cap \pi_2 = \emptyset$. Then $G$ has a semidirect decomposition $G = HN$, where $H$ is an abelian Hall $(\pi_1 \cup \pi_2)$-subgroup and $N$ is a normal Hall $(\pi_1 \cup \pi_2)'$-subgroup.*

*Proof:* Let $N = N_1 \cap N_2$. By Lemma 1.2: $|G\colon N| = |G\colon N_1| \cdot |G\colon N_2|$, so $N$ is a normal Hall $(\pi_1 \cup \pi_2)'$-subgroup. By Theorem 1.1 $G$ has a Hall $(\pi_1 \cup \pi_2)$-subgroup, $H$, so $G = HN$. Now $H \cong G/N$ is isomorphic to a subgroup of $G/N_1 \times G/N_2 \cong H_1 \times H_2$, whence $H$ is abelian. ∎

*Remark:* In the above proof the property *abelian* can be replaced by any other group property such that $G_1 \times G_2$ has it whenever $G_1$ and $G_2$ do.

COROLLARY 2.2: *Every group $G$ has a semidirect decomposition $G = HN$, where $H$ is an abelian Hall $\Lambda(G)$-subgroup and $N$ is a normal Hall $\Delta(G)$-subgroup. Furthermore: $N_G(C_G(H)) = C_G(H) = N_G(H)$.*

*Proof:* Apply Lemma 2.1 consecutively. By Theorem 1.3, for every $p \in \Lambda$ $G$ has a semidirect decomposition $G = PN$, where $P$ is an abelian Sylow (Hall) $p$-subgroup and $N$ is a normal Hall $p'$-subgroup. If $G = P_1N_1 = P_2N_2$ are two such decompositions for distinct $p_1, p_2 \in \Lambda$, then we proved that $G = H'N'$, where $H'$ is an abelian Hall $\{p_1, p_2\}$-subgroup and $N'$ is a normal Hall $\{p_1, p_2\}'$-subgroup. Assuming $G = P_3N_3$ is another decomposition, we can continue this process using Lemma 2.1, until we get the desired decomposition — $G = HN$.

Since $H$ is abelian, every Sylow subgroup $P$ of $H$ is characteristic in $H$ and therefore normalized by any element of $G$ which normalizes $H$. Conversely, if an element of $G$ normalizes every Sylow subgroup of $H$, it normalizes $H$. Similarly, an element of $G$ commutes with $H$ iff it commutes with every Sylow subgroup of $H$ and we get:

$$N_G(H) = \bigcap N_G(P) \overset{\text{def } \Lambda}{\overset{\downarrow}{=}} \bigcap C_G(P) = C_G(H)$$

where $P$ runs over all Sylow subgroups of $H$.

Since $H \trianglelefteq C_G(H)$, $H$ is a normal Hall subgroup of $C_G(H)$ and is thus characteristic, whence $N_G(C_G(H)) \leq N_G(H) = C_G(H)$ and equality holds. ∎

*Remark:* From the above we can easily deduce : $N = O_{\pi(G')}(G)$.

THEOREM 2.3: *For every group $G$, the vertices of $\Gamma^*(G)$ can be partitioned as $V^* = \Delta(G) \cup \Lambda(G)$ and the graph is of at least one of the following types:*

Type A: *$\Gamma^*(G)$ contains two (not neccessarily connected) vertex disjoint complete subgraphs — $\Gamma^*(G)\big|_\Delta$ and $\Gamma^*(G)\big|_\Lambda$.*

Type B: *$\Gamma^*(G)$ contains a complete subgraph — $\Gamma^*(G)\big|_\Delta$, to which every vertex is joined by an edge.*

*Proof:* Let $G = HN$ be the decomposition of Corollary 2.2, and let: $Z_H = Z(G) \cap H$. It is easy to see that $C_{G/Z_H}(gZ_H) = C_G(g)/Z_H$ thus: $\{\mathrm{Cl}_{G/Z_H}(gZ_H)\colon g \in G\} = \{\mathrm{Cl}_G(g)\colon g \in G\}$, so we may assume, without loss of generality, that $Z_H = 1$ and $Z(G) \leq N$. We recall that $\Gamma^*(G)\big|_\Delta$ is a complete graph, so let us deal with $p \in \Lambda$. Let $a \in G$ and $A = \mathrm{Cl}(a)$.

CLAIM 1: $(|A|, |N|) = 1$ if and only if $a \in Z(N)$.

*Proof:* $\Leftarrow$ Clearly $N \leq C_G(a)$, so $(|A|, |N|) = 1$.

$\Rightarrow$ Let $a = a_1 \times a_2$ be the decomposition of $a$ into the product of commuting elements, where $a_1$ is a $\Lambda$-element and $a_2$ is a $\Delta$-element. By Theorem 1.1 $a_2 \in N$, say $a_2 = n$, and $a_1 \in H^x$, say $a_1 = h^x$ with $h \in H$.

By Theorem 1.1 $N \leq C_G(a)$ and by Lemma 1.4 $C_G(a) = C_G(h^x) \cap C_G(n)$, so $n \in Z(N)$ and since $H^x \leq C_G(h^x)$ we have $H^xN \leq C_G(h^x)$, so $h^x \in Z(G)$ and $a_1 = h = 1$ follows. ∎

CLAIM 2: If $d^*(p, \Delta) > 1$ for some $p \in \pi(G)$, then $\Gamma^*(G)\big|_\Lambda$ is a complete graph.

*Proof:* Let $p$ be such a prime and let $p \mid |A|$, where $A = \mathrm{Cl}(a)$. Clearly $(|A|, |N|) = 1$, so by Claim 1 we can assume that $a = n \in Z(N)$.

Since $G = HN$ and $N \leq C_G(n)$ it follows that $C_G(n) = (C_G(n) \cap H)N$, and therefore $C_G(n) = C_H(n)N$.

Assume $C_H(n) > 1$. Then there is an element $h \in H \setminus Z(G)$ which commutes with $n$. But since $h$ and $n$ are commuting elements of relatively prime orders, we have by Lemma 1.4: $|\mathrm{Cl}(h)|, |\mathrm{Cl}(n)| \mid |\mathrm{Cl}(hn)|$. Now $1 \neq |\mathrm{Cl}(h)| \mid |N|$ and we get $d^*(p, \Delta) = 1$, a contradiction.

So $C_H(n) = 1$ whence: $\pi(A) = \pi(|H|) = \Lambda$ and $\Gamma^*(G)\big|_\Lambda$ is thus complete. ∎

To summarize: either $d^*(p, \Delta) = 1$ for every $p \in \Lambda$ or $\Gamma^*(G)\big|_\Lambda$ is complete and the theorem is proved. ∎

Consequently we get:

COROLLARY 2.4: *$\Gamma^*(G)$ (and so $\Gamma(G)$) has at most two connected components.*

COROLLARY 2.5: *If $\Gamma^*(G)$ is connected then $d^*(G) \leq 3$.*

## 3. $\Gamma(G)$

Using the settings of §2:

COROLLARY 3.1: *If $(|A|, |N|) = 1$ and $\Gamma(G)$ is connected, then there exist $C \in \mathrm{Con}(G)$ for which $(|A|, |C|) > 1$ and $(|N|, |C|) > 1$.*

COROLLARY 3.2: *If either $\Lambda$ or $\Delta$ consists of a single prime and $\Gamma(G)$ is connected, then $d^*(G) \leq 2$ whence, by Lemma 1.6, $d(G) \leq 3$.*

THEOREM 3.3: *If $\Gamma(G)$ is connected, then $d(G) \leq 3$.*

*Proof:* Let $A, B \in \mathrm{Con}(G)$ be non-central. Consider the following three possible cases:

I: $(|A|, |N|) > 1$ and $(|B|, |N|) > 1$.

Since $\Gamma^*(G)\big|_\Delta$ is complete we have $C \in \mathrm{Con}(G)$ such that: $(|A|, |C|) > 1$ and $(|B|, |C|) > 1$, so $d(A, B) \leq 2$.

II: $(|A|, |N|) = 1$ and $(|B|, |N|) > 1$.

By Corollary 3.1 we have $C \in \mathrm{Con}(G)$ with $d(A, C) = 1$ and $(|C|, |N|) > 1$ and by I $d(C, B) \leq 2$, so $d(A, B) \leq 3$.

III: $(|A|,|N|)=1$ and $(|B|,|N|)=1$.

Let $A=\mathrm{Cl}(a)$ and $B=\mathrm{Cl}(b)$. Assume, without loss of generality, that $o(a)=p^i$ and $o(b)=q^j$ where $p$ and $q$ are primes. By Claim 1 of Theorem 2.3 we can assume that $a,b\in Z(N)$.

If $p\neq q$, then by Lemma 1.4 $|A|,|B| \mid |\mathrm{Cl}(ab)|$, so $d(A,B)\leq 2$.

If $p=q$, then by Corollary 3.2 we can assume that there is a prime $r\in\Delta$, distinct from $p$. So there is an element $c\in N\setminus Z(G)$ with $o(c)=r^k$ and since $a,b\in Z(N)$, $c$ commutes with both. Thus by Lemma 1.4 we have the path:

$$A \underset{|A|}{\longleftrightarrow} \mathrm{Cl}(ac) \underset{|\mathrm{Cl}(c)|}{\longleftrightarrow} \mathrm{Cl}(bc) \underset{|B|}{\longleftrightarrow} B$$

and $d(A,B)\leq 3$. ∎

## 4. $\Gamma(G),\Gamma^*(G)$ — the disconnected case

THEOREM 4.1: *If $\Gamma^*(G)$ is disconnected, then $G$ is quasi-Frobenius with abelian kernel and complements.*

*Proof:* Since a group of *Type B* (in Theorem 2.3) is connected, $G$ is of *Type A* and $\Gamma^*(G)$ consists of two disconnected components:

$$\Gamma^*(G)\big|_\Lambda \quad\text{and}\quad \Gamma^*(G)\big|_\Delta$$

each of which is a complete subgraph.

Using the notation of Theorem 2.3 define:

$Z_G\equiv Z(G)$ , $Z\equiv Z(N)$ , $C_H\equiv C_G(H)$, and likewise assume that $Z(G)\leq N$. Every element of $C_H\cap Z$ commutes with both $H$ and $N$ and is therefore central. Thus $C_H\cap Z=Z_G$.

If $a\in G\setminus Z$, we showed (in Claim 1 of Theorem 2.3) that $(|\mathrm{Cl}(a)|,|N|)>1$ so $(|\mathrm{Cl}(a)|,|H|)=1$.

By Theorem 1.1, $C_G(a)$ also has a Hall $\Lambda$-subgroup and since $|H| \mid |C_G(a)|$, it is of the same order as $H$ and thus a conjugate of $H$, say $H^x\leq C_G(a)$, $x\in G$. It follows that $a\in C_G(H^x)$ which by Lemma 1.5 is a conjugate of $C_H$. We deduce:

$$(*)\qquad G=\left(\bigcup_{x\in G}(C_H^x\setminus Z_G)\right)\cup Z.$$

Therefore:

$$|G| \leq \frac{|G|}{|C_H|}(|C_H| - |Z_G|) + |Z| \quad \Rightarrow \quad \frac{|G||Z_G|}{|C_H|} \leq |Z|$$
$$\Rightarrow \quad |G| \leq \frac{|C_H||Z|}{|Z_G|} = |C_H Z| \leq |G|$$

and equality holds. Thus there is no redundancy in (∗) and therefore:

$$C_H^x \cap C_H = Z_G \quad \text{for every } x \notin N_G(C_H) = C_H \ ;$$
$$Z = Z_G \cup \left( G \smallsetminus \bigcup_{x \in G} C_H^x \right)$$

and taking these equations modulu $Z_G$, we see, by Lemma 1.8, that $G/Z_G$ is Frobenius with a complement $C \equiv C_H/Z_G$ and the kernel $K \equiv Z/Z_G$.

Now

$$|G/Z_G| = |C||K| = |HZ_G/Z_G||NZ_G/Z_G|,$$

and

$$(|C|, |K|) = (|HZ_G/Z_G|, |NZ_G/Z_G|) = 1,$$

so

$$\pi(G) = \pi(G/Z_G) = \pi(C) \uplus \pi(K) = \Lambda \uplus \Delta.$$

Obviously $HZ_G/Z_G \leq C$ and $K \leq NZ_G/Z_G$ so in order to show equality here we need only prove $\pi(C) = \Lambda$. Now $K$ is abelian so if $1 \neq k \in K$, then $C_{G/Z_G}(k) = K$ and if $k = xZ_G$, where $x \in Z$, we have that $C_G(x) \leq Z$. Since $Z$ is abelian we have equality, so $|\mathrm{Cl}(x)| = |G|/(|K||Z_G|) = |C|$. Thus $\Gamma^*(G)\big|_{\pi(C)}$ is a complete graph which contain $\Gamma^*(G)\big|_{\Lambda}$, whence $\pi(C) = \Lambda$. We conclude that $C = HZ_G/Z_G$ and $K = NZ_G/Z_G$ and hence both $C_H = HZ_G$ and $Z$ are abelian, as required. ∎

Now it follows from Lemma 1.6 that:

COROLLARY 4.2: *If $\Gamma(G)$ is disconnected, then $G$ is quasi-Frobenius with abelian kernel and complements.*

*Remark:* If $G$ is quasi-Frobenius with abelian kernel and complements, then obviously the graphs $\Gamma(G)$ and $\Gamma^*(G)$ are disconnected, and if $G = A \times G_1$, where $A$ is abelian, then: $G/Z(G) = (A \times G_1)/(A \times Z(G_1)) \cong G_1/Z(G_1)$. So in general (without assuming anything about abelian factors) it is true that $\Gamma(G)$ (equivalently $\Gamma^*(G)$) is disconnected if and only if $G$ is quasi-Frobenius with abelian kernel and complements.

## References

[1] Edward A. Bertram, Marcel Herzog and Avinoam Mann, *On a graph related to conjugacy classes of groups*, Bulletin of the London Mathematical Society **22** (1990), 569–575.

[2] David Chillag, Marcel Herzog and Avinoam Mann, *On the diameter of a graph related to conjugacy classes of groups*, Bulletin of the London Mathematical Society **25** (1993), 255–262.

[3] Silvio Dolfi, *Arithmetical conditions on the length of the conjugacy classes of a finite group*, to appear.

[4] Daniel Gorenstein, *Finite Groups*, Harper's Series in Modern Mathematics, Harper & Row, New York, 1968.

[5] Noboru Ito, *On finite groups with given conjugate types, I*, Nagoya Mathematical Journal **6** (1953), 17–28.

[6] L. S. Kazarin, *On groups with isolated conjugacy classes*, Izvestiya VUZ Matematika **25** (1981), 40–45.

[7] Derek J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1982.